A Shear Lag Analysis for Composite Box Girders with Deformable Connectors

V. Křístek

A method is proposed for shear lag analysis which can be applied to steel-concrete composite box girders. The proposed method uses harmonic analysis and allows the determination of shear lag effects from simple calculations so that the method is regarded as a design aid. The character of the method can illustrate the influence of certain key parameters upon the extent of the shear lag effect.

Keywords: shear lag, composite girder, stress distribution, harmonic analysis.

1 Introduction

The effects of shear lag can cause a significant increase in the longitudinal stresses developed in steel box girders. Previous investigations have shown that the extent of shear lag within a flange plate is dependent on the ratio between the axial stiffness and the shear stiffness of the plate. The introduction of longitudinal stiffeners increases the axial stiffness without changing the shear stiffness so that there is a consequent increase in shear lag. Stiffeners are, of course, introduced to increase the resistance of the compressed flange to buckling. It has been proven in [1] that it is far more advantageous, from the point of view of shear lag, if the flange plate is stiffened with a layer of concrete that is made to act compositely with the steel plate (Fig.1a). The necessary composite action can be achieved by means of shear studs welded to the steel plate.

Among many applications of composite arrangements, the case of increasing the load carrying capacity of an existing steel box girder bridge may be mentioned as a special example. The bottom flange plate in the hogging moment regions over the internal supports of a continuous girder is particularly susceptible to the effects of shear lag. The most obvious way of strengthening these regions is to weld on more longitudinal stiffeners in the compression zone of the bottom flange. This will increase the buckling resistance of the flange, but it will also accentuate the shear lag problem. An alternative method of strengthening an existing bridge girder is to add a concrete layer to the compression flange so that it acts compositely with the steel (Fig. 1b), which will increase the buckling resistance while also controlling the shear lag effect. Although the method is applicable primarily for strengthening an existing bridge, it may well provide an economic alternative in the design of a new box girder.

A perfect connection between the steel flange and the concrete layer exists, however, only theoretically. Although there certainly will be an intention to benefit from full composite interaction, the studs placed at regular distances, which are commonly used as connectors at the present time, exhibit some unavoidable deformability.

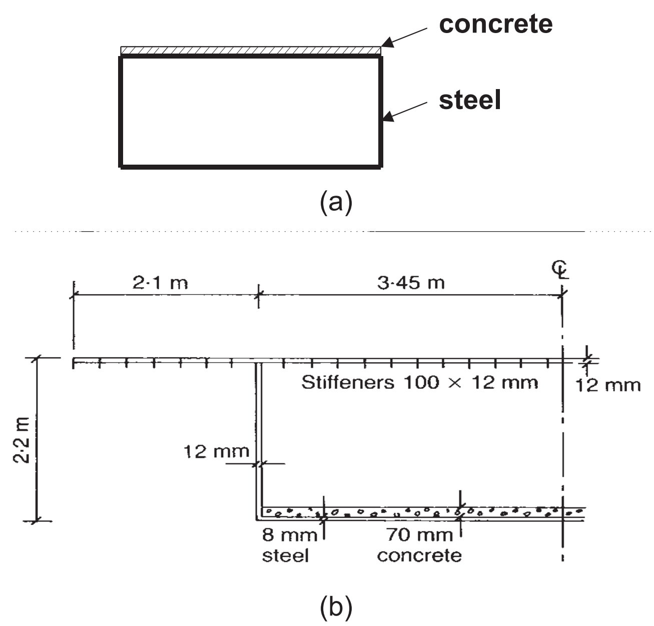

Fig. 1: (a) Steel-concrete composite girder – simplified form of cross section, (b) concrete layer added to the compression bottom flange plate in the hogging moment regions

2 Governing equations

Shear flows ${}_sq$ and ${}_cq$, and normal forces ${}_sn_x$ and ${}_cn_x$ per unit width act on a typical element of the steel flange sheet or the concrete layer, respectively (see Fig. 2).

The equations governing the equilibrium in the longitudinal direction are:

for the steel sheet (Fig. 2a)

$$\frac{\partial\, {}_sn_x}{\partial x} + \frac{\partial\, {}_sq}{\partial y} + f = 0, \tag{1}$$

for the concrete layer (Fig. 2b)

$$\frac{\partial\, {}_cn_x}{\partial x} + \frac{\partial\, {}_cq}{\partial y} - f = 0, \tag{2}$$

in which f is the shear acting in the longitudinal direction at the interface between the steel flange and the concrete layer.

If the contribution of small traverse forces to the strains is neglected, it may be written:

$${}_s\varepsilon_x = \frac{\partial\, {}_su}{\partial x} = \frac{{}_sn_x}{t_s E_s}, \tag{3}$$

$${}_c\varepsilon_x = \frac{\partial\, {}_cu}{\partial x} = \frac{{}_cn_x}{t_c E_c}, \tag{4}$$

$${}_s\varepsilon_y = -\nu_s\, {}_s\varepsilon_x = -\nu_s \frac{{}_sn_x}{t_s E_s}, \tag{5}$$

$${}_c\varepsilon_y = -\nu_c\, {}_c\varepsilon_x = -\nu_c \frac{{}_cn_x}{t_c E_s}, \tag{6}$$

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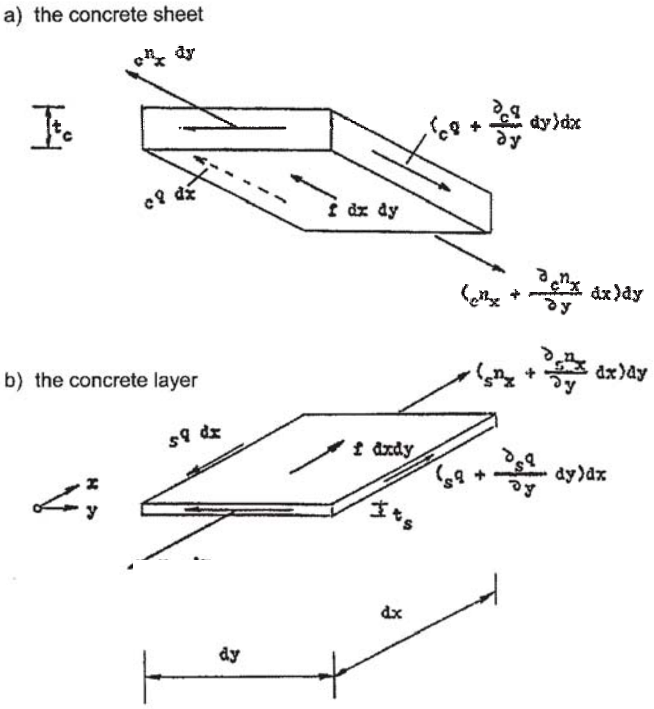


Fig. 2: Equilibrium conditions in the longitudinal direction

$$\gamma_s = \frac{{}_sq}{t_s G_s}, \tag{7}$$

$$\gamma_c = \frac{{}_cq}{t_c G_c}, \tag{8}$$

where t_s and t_c are the thicknesses of the steel flange and the concrete layer, respectively, ε_x and ε_y are the direct strains in the longitudinal and transverse directions, respectively, and γ are the shear strains. E, G and ν represent Young's moduli, the shear moduli and Poisson's ratios, respectively; u are the longitudinal displacements.

The general form of the condition of compatibility is as follows:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma}{\partial x\,\partial y}. \tag{9}$$

Substituting the strains from Eqs. (3)–(8) it is obtained:

for the steel:

$$\frac{\partial^2 {}_sn_x}{\partial y^2} - \nu_s \frac{\partial^2 {}_sn_x}{\partial x^2} = 2(1+\nu_s)\frac{\partial^2 {}_sq}{\partial x\,\partial y}, \tag{10}$$

for the concrete layer:

$$\frac{\partial^2 {}_cn_x}{\partial y^2} - \nu_c \frac{\partial^2 {}_cn_x}{\partial x^2} = 2(1+\nu_c)\frac{\partial^2 {}_cq}{\partial x\,\partial y}. \tag{11}$$

Substituting for the shear flows ${}_sq$ and ${}_cq$ from equations (1) and (2):

$$\frac{\partial^2 {}_sn_x}{\partial y^2} + (2+\nu_s)\frac{\partial^2 {}_sn_x}{\partial x^2} + 2(1+\nu_s)\frac{\partial f}{\partial x} = 0, \tag{12}$$

$$\frac{\partial^2 {}_cn_x}{\partial y^2} + (2+\nu_c)\frac{\partial^2 {}_cn_x}{\partial x^2} - 2(1+\nu_c)\frac{\partial f}{\partial x} = 0. \tag{13}$$

It may be assumed that the shear f acting between the steel sheet and the concrete layer, being provided by deformable connectors, is proportional to the mutual longitudinal slip which occurs at the interface between the two components, i.e.

$$f = k({}_su - {}_cu) \tag{14}$$

where k is the connector stiffness.

Eq. (14) may be written in the form:

$$\frac{\partial f}{\partial x} = k\left(\frac{\partial\,{}_su}{\partial x} - \frac{\partial\,{}_cu}{\partial x}\right) = k\left(\frac{{}_sn_x}{t_s E_s} - \frac{{}_cn_x}{t_c E_c}\right). \tag{15}$$

The following Fourier series may express the searched functions:

$${}_sn_x = \sum {}_sN_j(y)\sin\frac{j\pi x}{L}, \tag{16}$$

$${}_cn_x = \sum {}_cN_j(y)\sin\frac{j\pi x}{L}, \tag{17}$$

$$f = \sum_{j=1}^{\infty} F_j(y)\cos\frac{j\pi x}{L}, \tag{18}$$

where L is the effective span-length.

Eqs. (12), (13) and (15) can be written in the form:

$${}_s\ddot{N}_j - \frac{j^2\pi^2}{L^2}(2+\nu_s)\,{}_sN_j - \frac{2j\pi}{L}(1+\nu_s)F_j = 0, \tag{19}$$

$${}_c\ddot{N}_j - \frac{j^2\pi^2}{L^2}(2+\nu_c)\,{}_cN_j - \frac{2j\pi}{L}(1+\nu_c)F_j = 0, \tag{20}$$

$$\frac{j\pi}{L}F_j + k\left(\frac{{}_sN_j}{t_s E_s} - \frac{{}_cN_j}{t_c E_c}\right) = 0, \tag{21}$$

in which ${}_s\ddot{N}_j = \dfrac{\partial^2 {}_sN_j}{\partial y^2}$, etc.

These relations represent a set of three equations for the unknown functions ${}_sN_j(y)$, ${}_cN_j(y)$ and $F_j(y)$, which can be adjusted to the following system of two differential equations

$${}_s\ddot{N}_j + a_j\,{}_sN_j - c_j\,{}_cN_j = 0, \tag{22}$$

$${}_c\ddot{N}_j + b_j\,{}_cN_j - d_j\,{}_sN_j = 0, \tag{23}$$

where

$$\begin{aligned} a_j &= -\frac{j^2\pi^2}{L^2}(2+\nu_s) + \frac{2k(1+\nu_s)}{t_s E_s}, \\ b_j &= -\frac{j^2\pi^2}{L^2}(2+\nu_c) + \frac{2k(1+\nu_c)}{t_c E_c}, \\ c_j &= \frac{2k(1+\nu_s)}{t_c E_c}, \\ d_j &= \frac{2k(1+\nu_c)}{t_s E_s}. \end{aligned} \tag{24}$$

It follows from Eq. (22) that

$${}_cN_j = \frac{1}{c_j}({}_s\ddot{N}_j + a_j\,{}_sN_j), \tag{25}$$

which substituted into Eq. (23) allows to obtain a differential equation of the fourth order

$$\frac{\mathrm{d}^4{}_sN_j(y)}{\mathrm{d}y^4} + A_j\frac{\mathrm{d}^2{}_sN_j(y)}{\mathrm{d}y^2} + B_j^2\,{}_sN_j(y) = 0, \qquad (26)$$

whose coefficients are

$$A_j = -\frac{j^2\pi^2}{L^2}(4+\nu_s+\nu_c) + 2k\left[\frac{(1+\nu_s)}{t_s E_s} + \frac{(1+\nu_c)}{t_c E_c}\right],$$
$$B_j^2 = \frac{j^2\pi^2}{L^2}\left\{\frac{j^2\pi^2}{L^2}(2+\nu_s)(2+\nu_c) + \right. \qquad (27)$$
$$\left. +2k\left[\frac{(1+\nu_s)(2+\nu_c)}{t_s E_s} + \frac{(1+\nu_c)(2+\nu_s)}{t_c E_c}\right]\right\}.$$

The general solution of (26), if the case of complex roots of the characteristic equation is assumed, is

$${}_sN_j(y) = C_{1,j}\,P_{1,j}(y) + C_{2,j}\,P_{2,j}(y) + $$
$$+C_{3,j}\,P_{3,j}(y) + C_{4,j}\,P_{4,j}(y), \qquad (28)$$

where

$$P_{1,j}(y) = \sinh\xi_j y \sin\eta_j y$$
$$P_{2,j}(y) = \cosh\xi_j y \cos\eta_j y \qquad (29)$$
$$P_{3,j}(y) = \cosh\xi_j y \sin\eta_j y$$
$$P_{4,j}(y) = \sinh\xi_j y \cos\eta_j y$$

and

$$\xi_j = \sqrt{\frac{B_j}{2} - \frac{A_j}{4}}, \quad \eta_j = \sqrt{\frac{B_j}{2} + \frac{A_j}{4}}.$$

The amplitude function ${}_cN_j(y)$, according to Eq. (25), is determined as

$${}_cN_j(y) = \frac{1}{c_j}\left\{C_{1,j}[r_jP_{1,j}(y) + s_jP_{2,j}(y)] + \right.$$
$$+C_{2,j}[r_jP_{2,j}(y) - s_jP_{1,j}(y)] + \qquad (30)$$
$$+C_{3,j}[r_jP_{3,j}(y) + s_jP_{4,j}(y)] + $$
$$\left. +C_{4,j}[r_jP_{4,j}(y) - s_jP_{3,j}(y)]\right\},$$

in which

$$r_j = \xi_j^2 - \eta_j^2 + a_i,$$
$$s_j = 2\xi_j\eta_j. \qquad (31)$$

3 Boundary and loading conditions

Shear lag analysis is carried out for loads, placed symmetrically on the girder cross-section. Thus, assuming the origin of the traverse co-ordinate y to be taken at the mid-width of the flange, i. e. at the axis of symmetry, then, because of the symmetry

$$C_{3,j} = C_{4,j} = 0 \qquad (32)$$

so that from equations (28) and (30), the distributions across the flange width of the normal forces in the steel flange and in the concrete layer are governed by:

$${}_sN_j(y) = C_{1,j}P_{1,j}(y) + C_{2,j}P_{2,j}(y) \qquad (33)$$

$${}_cN_j(y) = \frac{1}{c_j}\left\{C_{1,j}[r_jP_{1,j}(y) + s_jP_{2,j}(y)] + \right. \qquad (34)$$
$$\left. +C_{2,j}[r_jP_{2,j}(y) - s_jP_{1,j}(y)]\right\}.$$

The amplitude function governing the distribution of the shear at the interface between the steel and concrete can be expressed from equation (21) as

$$F_j(y) = \frac{kL}{j\pi}\left\{C_{1,j}\left\langle\frac{1}{c_j t_c E_c}[r_jP_{1,j}(y) + s_jP_{2,j}(y)] - \right.\right.$$
$$\left. -\frac{P_{1,j}(y)}{t_s E_s}\right\rangle + \qquad (35)$$
$$+C_{2,j}\left\langle\frac{1}{c_j t_c E_c}[r_jP_{2,j}(y) - s_jP_{1,j}(y)] - \right.$$
$$\left.\left. -\frac{P_{2,j}(y)}{t_s E_s}\right\rangle\right\}.$$

It is seen that also this distribution is symmetrical about the flange mid-width.

The values of the remaining constants $C_{1,j}$ and $C_{2,j}$ can be determined from the shear loading conditions at the edges of the steel flange and the concrete layer.

Combining equations (1), (16), (18):

$$\frac{\partial\,{}_sq}{\partial y} = -\frac{\partial\,{}_sn_x}{\partial x} - f = \sum_{j=1}^{\infty}\left[\frac{j\pi}{L}\,{}_sN_j(y) + F_j(y)\right]\cos\frac{j\pi x}{L} = $$
$$= \sum_{j=1}^{\infty}\left\{C_{1,j}\left\langle\left[\frac{j\pi}{L} + \frac{kL}{j\pi}\left(\frac{r_j}{c_jt_c E_c} - \frac{1}{t_s E_s}\right)\right]P_{1,j}(y) + \right.\right.$$
$$\left. +\frac{kLs_j}{j\pi c_j t_c E_c}P_{2,j}(y)\right\rangle + \qquad (36)$$
$$+C_{2,j}\left\langle\left[\frac{j\pi}{L} + \frac{kL}{j\pi}\left(\frac{r_j}{c_jt_c E_c} - \frac{1}{t_s E_s}\right)\right]P_{2,j}(y) - \right.$$
$$\left.\left. -\frac{kLs_j}{j\pi c_j t_c E_c}P_{1,j}(y)\right\rangle\right\}\cos\frac{j\pi x}{L}$$

so that (by integrating with respect to y) the shear flow in the steel flange at any point may be expressed as:

$${}_sq(x,y) = \sum_{j=1}^{\infty}\frac{1}{\xi_j^2+\eta_j^2}\left\{C_{1,j}\left\langle\left[\frac{j\pi}{L} + \frac{kL}{j\pi}\left(\frac{r_j}{c_jt_c E_c} - \frac{1}{t_s E_s}\right)\right]\times\right.\right.$$
$$\times[\xi_j\,P_{3,j}(y) - \eta_j\,P_{4,j}(y)] + $$
$$\left. +\frac{kLs_j}{j\pi c_j t_c E_c}[\xi_j\,P_{4,j}(y) + \eta_j\,P_{3,j}(y)]\right\rangle + $$
$$+C_{2,j}\left\langle\left[\frac{j\pi}{L} + \frac{kL}{j\pi}\left(\frac{r_j}{c_jt_c E_c} - \frac{1}{t_s E_s}\right)\right]\times\right. \qquad (37)$$
$$\times[\xi_j\,P_{4,j}(y) + \eta_j\,P_{3,j}(y)] - $$
$$\left.\left. -\frac{kLs_j}{j\pi c_j t_c E_c}[\xi_j\,P_{3,j}(y) - \eta_j\,P_{4,j}(y)]\right\rangle\right\}\cos\frac{j\pi x}{L} = $$
$$= -\sum_{j=1}^{\infty}[C_{1,j}\,Z_{1,j}(y) + C_{2,j}\,Z_{2,j}(y)]\cos\frac{j\pi x}{L}.$$

Similarly, the shear flow in the concrete layer, combining equations (2), (17) and (18), is governed by the following relation:

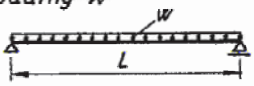
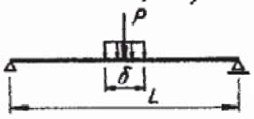
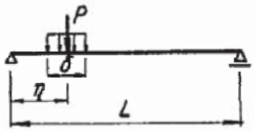
$$\frac{\partial _c q}{\partial y} = -\frac{\partial _c n_x}{\partial x} + f = -\sum_{j=1}^{\infty}\left[\frac{j\pi}{L}\, {}_cN_j\,(y) - F_j\,(y)\right]\cos\frac{j\pi x}{L} =$$

$$= -\sum_{j=1}^{\infty}\left\{C_{1,j}\left\langle\left(\frac{j\pi}{Lc_j} + \frac{kL}{j\pi c_j\, t_c\, E_c}\right)[r_j\, P_{1,j}(y) + s_j\, P_{2,j}(y)] + \right.\right.$$

$$\left. + \frac{kL}{j\pi\, t_s\, E_s} P_{1,j}(y)\right\rangle +$$

$$+ C_{2,j}\left\langle \frac{j\pi}{Lc_j} + \frac{kL}{j\pi c_j\, t_c\, E}\right.$$

Table 1: Values of the coefficient $Q_{e,j}$ for different tapes of loading

Type of loading	$Q_{e,j}$ Section 3.1.4 (Eq. (3.24))	$Q_{e,j}$ Section 3.1.5
Uniformly distributed loading w 	$2\dfrac{Lw\bar{t}be}{j^2\pi^2 I}$	$2\dfrac{Lw(tb+\Sigma A_s)e}{j^2\pi^2 I}$
Force P distributed over length δ (symmetrical about mid-span) 	$2\dfrac{PL\bar{t}be}{\delta j^2\pi^2 I}\sin\dfrac{j\pi}{2}\sin\dfrac{j\pi\delta}{2L}$	$2\dfrac{PL(tb+\Sigma A_s)e}{\delta j^2\pi^2 I}\sin\dfrac{j\pi}{2}\sin\dfrac{j\pi\delta}{2L}$
Force P distributed over length δ (at general position) 	$2\dfrac{PL\bar{t}be}{\delta j^2\pi^2 I}\sin\dfrac{j\pi\eta}{L}\sin\dfrac{j\pi\delta}{2L}$ $\left(\text{for } \dfrac{\delta}{2} \le \eta \le L-\dfrac{\delta}{2}\right)$	$2\dfrac{PL(tb+\Sigma A_s)e}{\delta j^2\pi^2 I}\sin\dfrac{j\pi\eta}{L}\sin\dfrac{j\pi\delta}{2L}$

Knowing the amplitudes ${}_sN_j$ and ${}_cN_j$, the values of the longitudinal normal forces per unit width ${}_sn_x(x,y)$ and ${}_cn_x(x,y)$ may be determined from equations (16) and (17) for any position on the flange. Also the shear flows ${}_sq(x,y)$ and ${}_cq(x,y)$ at any point may be determined from equations (37) and (39).

To evaluate the forces taken by the studs, the shear f acting at the interface between the steel flange and the concrete layer is to be determined according to equation (18). The amplitude function $F_j(y)$, describing the distribution of the shear across the flange width, is determined – knowing constants $C_{1,j}$ and $C_{2,j}$ – by equation (35).

5 Summary of calculations

For any particular girder with composite flanges, the first step in the calculation of the shear lag effect is to determine the value of coefficient $Q_{e,j}$ from equation (42). The value $Q_{e,j}$ is then substituted into the right-hand side of equations (45) to give the values of constants $C_{1,j}$ and $C_{2,j}$, and, finally, for any harmonic the amplitudes of all the functions are required. These, in turn, are substituted into equations (16), (17), (18), (37) and (39) to give the normal forces per unit width, the shear acting at the interface between the steel and concrete components, and the shear flows at any position on the composite flange.

The corresponding value of the longitudinal stress in the steel component of the flange is then given by:

$$\sigma_{sx}(x,y) = \frac{{}_sn_x(x,y)}{t_s} \tag{46}$$

and the longitudinal stress in the concrete layer is obtained as:

$$\sigma_{cx}(x,y) = \frac{{}_cn_x(x,y)}{t_c}. \tag{47}$$

Should the shear stress values also be required, then, having evaluated the shear flows ${}_sq(x,y)$ and ${}_cq(x,y)$ at any point, the shearing stress in the steel is obtained as:

$$\tau_{sx}(x,y) = \frac{{}_sq(x,y)}{t_s} \tag{48}$$

and the shear stress in the concrete layer is given by:

$$\tau_{cx}(x,y) = \frac{{}_cq(x,y)}{t_c}. \tag{49}$$

6 Conclusions

This paper has described the development of an approximate analytical method for analysing the stress distribution in the flanges of composite steel-concrete beams with deformable connectors. Its primary advantage is the closed form of the results obtained and its ease of application. The method is also very suitable for parametric studies investigating the influences of various arrangements, and for optimisation studies.

To conclude, it should be noted that – besides the mechanical effects – the thermal effects can also play an important role in the structural performance of steel-concrete composite beams, see, e.g., [2].

7 Acknowledgment

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Prof. Ing. Vladimír Křístek, DrSc.
phone: +420 224 353 875
e-mail: kristek@fsv.cvut.cz

Department of Concrete Structures and Bridges

Czech Technical University in Prague
Faculty of Civil Engineering
Thákurova 7
Praha 6, Czech Republic

