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EXTERNAL ROLLING OF A POLYGON ON CLOSED CURVILINEAR PROFILE

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Abstract. The rolling of a flat figure in the form of an equilateral polygon on a curvilinear profile is considered. The profile is periodic. It is formed by a series connection of an arc of a symmetrical curve. The ends of the arc rely on a circle of a given radius. The equation of the curve, from which the curvilinear profile is constructed, is found. This is done provided that the centre of the polygon, when it rolls in profile, must also move in a circle. Rolling occurs in the absence of sliding. Therefore, the length of the arc of the curve is equal to the length of the side of the polygon. To find the equations of the curve of the profile, a first-order differential equation is constructed. Its analytical solution is obtained. The parametric equations of the curve are obtained in the polar coordinate system. The limits of the change of an angular parameter for the construction of a profile element are found. It is a part of the arc of the curve. According to the obtained equations, curvilinear profiles with different numbers of their elements are constructed.

Keywords: Equilateral polygon, curvilinear profile, external rolling, differential equation, centroids.

1. Introduction

Some flat figures, including polygons that can be rolled on a curvilinear periodic profile without sliding, are considered in [1]. The profile is formed by equal symmetrical curvilinear elements of their serial connection so that the ends of the elements abut on a straight line. When rolling a polygon on such profile, its centre moves in a straight line. Constructing a closed profile in which curvilinear elements touch a circle is important for the design of centroids of non-circular wheels. When rolling a polygon on such profile, its centre moves in a circle. If both centres (centre of the curvilinear profile and centre of the polygon) are stationary, then you can roll these figures while rotating around their centres. One centroid will be a polygon, the other will be a closed profile. Many works are devoted to the study of the rolling of flat figures one by one. There are common examples of rolling a straight line segment along a curve and vice versa - rolling a curvilinear profile along a straight line. For the first case, the rolling of a straight line in a circle is the classic one, as a result of which the point of the line describes the evolvent of a circle, and for the second case, the rolling of a circle along a straight line, as a result of which a circle point describes a cycloid [2]. In [3], information on the rolling of second-order curves along a straight line is given. The trajectory of focus in such rolling is known curves. The formation of flat curves by given kinematic parameters is considered in [4]. The basics of designing non-circular wheels for gears are given in [5]. Geometric modelling of a centroid of non-circular wheels was further developed in the works [6–8]. The use of non-circular wheels in gears has been considered in [9–17], in chain drives in a monograph [18]. The purpose of the article is to develop an analytical description of a curvilinear closed profile, in which an equilateral polygon will be rolled without sliding and its centre will move in a circle of a given radius.

2. Material and method

Consider the rolling of a polygon by the example of a square. We need to find a form of a flat profile in which the square will be rolled without sliding, and its centre will move in a circle of radius r (Fig. 1,a). Rolling of a square can be considered as an example of rolling of a triangle, the basis of which is the side of the square, and the altitude - the distance $AC=a$ from its centre to the side. In the initial position, the altitude of the AC is located on the Ox axis (Fig. 1,a). When rolling a square, its centre moves in a circle of a radius r, and the side rolls along the curve, that will be found. The moment comes when the point of contact of the square with the curve becomes its vertex. The diagonal $A'C'$ in this position passes through the origin of coordinates - the point O. It is obvious that the length s of the arc AA' is equal to the length of half of the side of the square (Fig. 1,a). When rolling, the side of the square is tangent to the curve, and in the position where the point of contact is its vertex, both sides of the square touch the curves. With further rolling, the process is repeated. In such a way that a rectilinear plane profile will consist of equal arcs that intersect on a circle of radius r_o at right angle (Fig. 1,a).

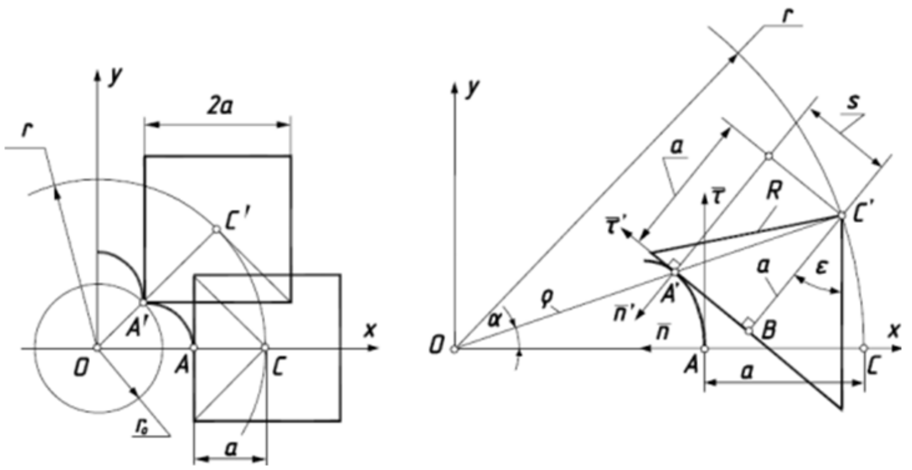


Figure 1. Graphic illustrations for rolling a square along a curvilinear contour: a) schematic representation of the two positions of the square when it is rolling; b) the current point of contact *A'* in the polar coordinate system.

In Fig. 1,a, it is shown that in the position when the point of contact with the curve is the vertex of the square, its centre is on the radius-vector *OC'*. This applies to any current point of contact of the side of the square with the curve. Let the triangle, which is the fourth part of the square, touch the curve at the current point *A'* (Fig. 1,b). As it is rolling without sliding, the current point of contact *A'* can be considered as the instantaneous centre of the rotation of the segment *A'C'* around it. In this case, the direction of the velocity of the point *C'* must be perpendicular to the segment *A'C'*. However, on condition that the point *C'* moves in a circle of a radius r, that is, it must be perpendicular to the radius-vector *OC'*. Thus, the centre of the square *C'* (or the vertex of the corresponding triangle, whose base is the side of the square) and the point of contact of the side of square *A'* are located on a common radius-vector that starts at the origin. Due to this, the equation of the curve of the profile is conveniently considered in the polar coordinate system.

Denote the distance from the origin of coordinates to the point of contact $OA'=\rho$ (Fig. 1,b), where ρ is a function of the angle α: $\rho=\rho(\alpha)$. The constant distance r is the sum of two segments of variable lengths: $r=\rho+A'C'$. The length of the hypotenuse *A'C'* can be found from the right triangle *A'C'B*. We find the expressions of the lengths of the legs *A'B* and *BC'* for the current tangency point *A'*. For a better understanding of the rolling process, consider the moving coordinate system - the accompanying trihedral of the curve of the profile, in which the unit vector $\overline{\tau}$ is tangent to it, the unit vector $\overline{n}$ is perpendicular, and the unit vector of the binormal $\overline{b}$ is projected to a point. The vertex of the trihedral is point *À* in its initial position, with the altitude *AC* lying on the *Ox* axis (Fig. 1, a, b), which coincides with the unit normal vector $\overline{n}$ (Fig. 1, b). When rolling the triangle, the point of contact *A'*, which is the vertex of the trihedral, moves along the curve and unit vector $\overline{\tau}$ remains tangent to the curve. In the initial position $\overline{\tau n}$ of the trihedral, points *A* and *B* coincided, and the altitude a of the triangle coincided with the unit vector$\overline{n}$. When rolling a triangle along the curve, the trihedral occupies a new position $\overline{\tau' n'}$ with the vertex at point *A'* (Fig. 1,b). The coordinates of the point *C'* in its system are as follows. The segment *A'B* is equal to the length of the arc *AA'* of the curve: $AA'=A'B=s$. The altitude a of the triangle when it is rolling remains parallel to the principal normal$\overline{n'}$. So the length of the segment *A'C'* is determined by the Pythagorean theorem: $A'C' = \sqrt{s^2+a^2}$. The expression $r=\rho+A'C'$ can be written as follows:

$$r = \rho + \sqrt{s^2 + a^2}. \tag{1}$$

Let's solve equation (1) for s:

$$s = \sqrt{(r-\rho)^2 - a^2}. \tag{2}$$

Let's write the parametric equations of the curve of the profile in the polar coordinate system:

$$\begin{aligned} x &= \rho\cos\alpha\,; \\ y &= \rho\sin\alpha\,. \end{aligned} \tag{3}$$

Let's find the expression of the arc length s of the curve (3). To do this, we define its first derivatives:

$$\begin{aligned} x' &= \rho'\cos\alpha - \rho\sin\alpha; \\ y' &= \rho'\sin\alpha + \rho\cos\alpha\,. \end{aligned} \tag{4}$$

By the known formula we write:

$$\frac{ds}{d\alpha} = \sqrt{x'^2 + y'^2} = \sqrt{\rho^2 + \rho'^2}. \tag{5}$$

The derivative of arc s can be found by a differentiation of expression (2):

$$\frac{ds}{d\alpha} = -\frac{\rho'(r-\rho)}{\sqrt{(r-\rho)^2 - a^2}}. \tag{6}$$

We equate expressions (5) and (6) and solve for ρ':

$$\frac{d\rho}{d\alpha} = \frac{\rho}{a}\sqrt{(r-\rho)^2 - a^2}. \tag{7}$$

The differential equation of first order (7) is obtained on the basis of the equality of the arcs of the profile curve and the side of the square that rolls on it without sliding. The altitude a of the triangle can be found through the angle $\varepsilon: \ a = R \cdot \cos\varepsilon$, where R is the length of the side of the triangle that is equal to the radius of the circle circumscribed about the square. Let's extend this expression to a polygon with an arbitrary number of sides n. The angle ε, in this case, will depend on the number of sides of the polygon: $\varepsilon=\pi/n$. The differential equation (7) for a polygon with an arbitrary number of sides n, being inscribed in a circle of radius R, will be written:

$$\frac{d\rho}{d\alpha} = \frac{\rho}{R\cos(\pi/n)}\sqrt{(r-\rho)^2 - R^2\cos^2(\pi/n)}. \quad (8)$$

Differential equation (8) has an analytical solution. On a condition that $\alpha=0$ $\rho=r\text{-}a=r\text{-}R \cdot \cos(\pi/n)$, we find the corresponding value of the constant of integration. With consideration of this constant, the solution of equation (8) takes the final form:

$$\rho = \frac{r^2 - R^2\cos^2(\pi/n)}{r + R\cos(\pi/n)\cosh\left(\alpha\sqrt{\frac{r^2}{R^2\cos^2(\pi/n)} - 1}\right)}. \quad (9)$$

Substitution (9) in (3) will give parametric equations of the curve. We need a limited arc to construct a profile. The magnitude of this arc is due to the minimum value of the radius-vector $\rho=r_o$ (Fig. 1a). Inversely, $r_o=r\text{-}A'C'=r\text{-}R$. Let's substitute in (9) $\rho=r\text{-}R$ and solve for α:

$$\alpha_0 = \pm\frac{R\cos(\pi/n)}{\sqrt{r^2 - R^2\cos^2(\pi/n)}}\operatorname{Arc}\cosh\left(\frac{r - R\cos^2(\pi/n)}{(r-R)\cos(\pi/n)}\right). \quad (10)$$

The wanted arc of the curve at given values of r, R and n is constructed according to equations (3) taking into account (9) when the angle α changes within $\alpha = -\alpha_{\hat{\imath}} \ldots \alpha_{\hat{\imath}}$. However, in this case, we will not be able to place the required number of arcs so that a closed profile is obtained. If we want to construct a profile of four arcs, then the angle $\alpha_{\hat{\imath}}=\pm\pi/4$ (this case is shown in Fig. 1,a). The measure of the angle α_o is determined by dividing the number π by the number of arcs. Hence, for a given number of arcs of a profile and the number of sides of a polygon of two radii r and R, we can only specify one of them, since the angle $\alpha_{\hat{\imath}}$ will also be given. Equation (10) cannot be solved with respect to one of the radii r or R, so numerical techniques must be used.

3. Result

It should be noted that the curvilinear profile can consist of one arc. If the polygon is a square, then at $\alpha_i = \pm\pi$, $n = 4$, $r = 100$, we find: $R = 95.28$. In Fig. 2, we can see a curvilinear profile and a square in two positions on opposite sides. When the profile is completely rolled, only one side of the square contacts it, that is, the square makes quarter of a turn.

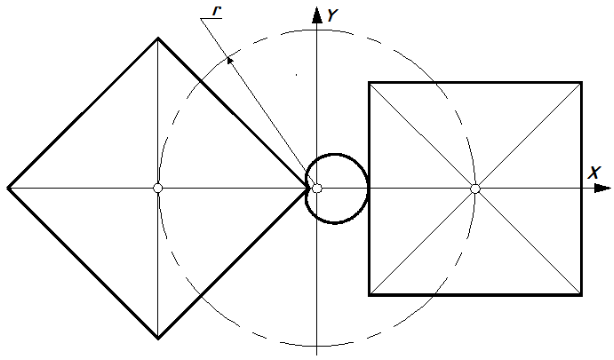


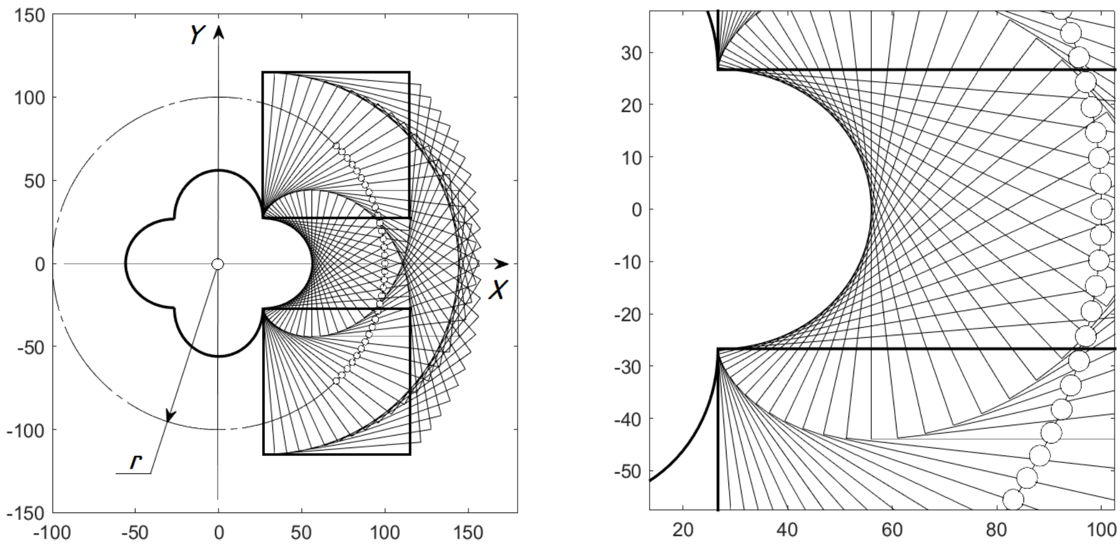
FIGURE 2. Curvilinear profile, with complete rolling during which the square makes quarter of a turn.

If a curvilinear profile consists of four elements, then the square during one complete rolling of the profile makes one turn. Repeating the calculation at $\alpha_i = \pm\pi/4$, $n = 4$, $r = 100$, we find: $R = 62.27$. In Fig. 3, we can see a curvilinear profile of four elements and a set of positions of a square when it is rolled along one of the elements. Sequential movement of the centre of the square when it is rolled is shown by circles. In the extreme positions of a square, when its vertices are points of contact with the profile, the sides of the square are depicted thickened.

It should be noted that for the physical rolling of a polygon along a curvilinear profile, there is a limit on the number of its sides. The number of sides of a polygon cannot be less than four. This is explained as follows. When rolling a polygon, its vertex describes a known curve - evolvent. Its property is that at the moment of detachment from the curve, the point of the straight that rolls on it (in our case the end of the side of the square) moves perpendicular to it. This can be seen from the enlarged fragment in Fig. 3, b. If the polygon was a triangle, then the angle between neighbouring elements would be 60° and physical rolling would be impossible. The considered approach allows constructing a curvilinear profile that would provide the required number of revolutions of the polygon at complete rolling along the profile. It is determined by the ratio of the number of profile elements to the number of sides of the polygon.

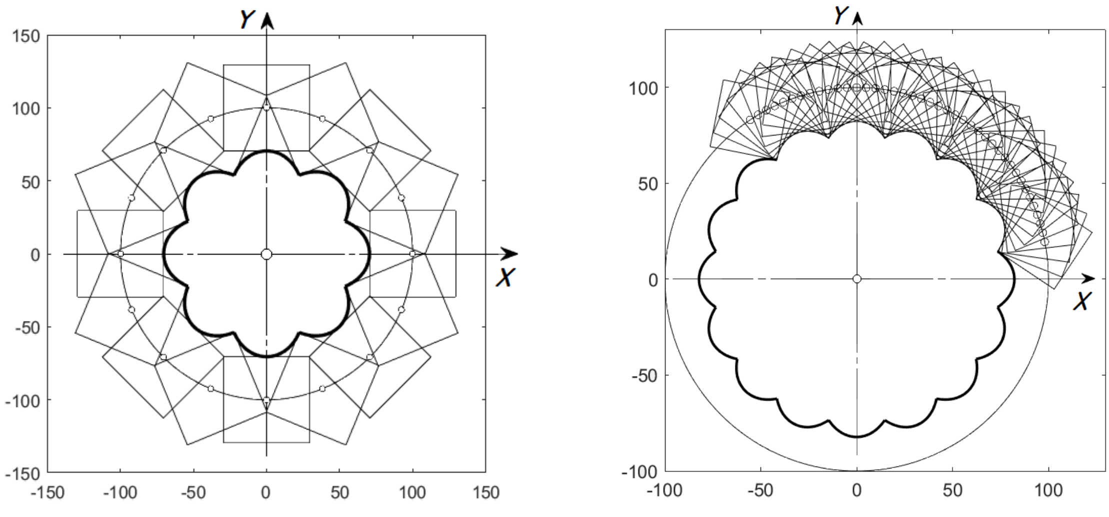
In Fig. 4,a, a profile consisting of 8 elements is constructed. At $r = 100$, the radius of the circumscribed circle is $R = 41.76$. When completely rolling along the profile, the square makes 2 turns. To ensure 4 turns, the profile must be 16 elements, with a radius of $R = 25.13$ (Fig. 4,b).

With an unlimited increase in the number n of the sides of a polygon, it transforms into a circle, and the radius-vector ρ into a constant value r–R, that is, $r_{\hat{\imath}}$ (Fig. 1,a). The number of turns of a circle of radius R, after a completed rolling along a circle of radius $r_{\hat{\imath}}$, is determined by the ratio of these radii.



(A). (B).

FIGURE 3. Curvilinear profile of four elements, when completely rolling along it, the square makes one turn: a) the set of positions of the square when it is rolling along the arc of the profile; b) enlarged fragment of the profile element



(A). (B).

FIGURE 4. Curvilinear profiles with different number of turns of a square: a) a square, when completely rolling along the profile, makes two turns; b) a square, when completely rolling along the profile, makes four turns.

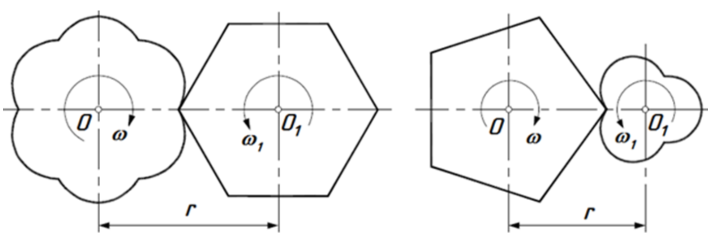


FIGURE 5. Polygons and curvilinear profiles with different numbers of sides and curvilinear elements at $r = 100$: a) a hexagon with a radius of the circumscribed circle $R = 54.93$ and a corresponding curvilinear profile with six elements; b) a pentagon with a radius of the circumscribed circle $R = 71.47$ and a corresponding curvilinear profile with three elements.

The correspondence of the number of sides of a polygon and elements of a curvilinear profile can be different (Fig. 5). Such figures can roll one by one with simultaneous rotation around fixed centres O and O_1 with angular velocities ω and ω_1, that is, they can serve as centroids for the design of non-circular gears [6–8]. The predetermined value r is the centre to centre distance.

When rotating one non-circular wheel at a constant angular velocity, the second will rotate at a variable angular velocity. This is due to the variable radius from the centres of the wheels to the point of the contact during the rotation. The radius-vector ρ changes from the maximum value at $\alpha = 0$ to the minimum $\rho = r_i = r - R$ (Fig. 1,a, points A and A'). This distance difference is the difference between the maximum and minimum values of the distance from the axis of rotation of the polygon to the point of contact with the curvilinear profile. It decreases as the number of sides of a polygon increases as well as increase the number of curvilinear elements of the profile that can be traced by the example of the square in Fig. 2 - 4.

4. Conclusions and prospects for further research

An analytical description of the rolling of a polygon along a curvilinear closed profile can be used to design non-circular gear wheels. The number of sides of a polygon must be at least four in order to be physically rolling. To design a curvilinear centroid, it is necessary to specify the distance from centre to centre, the number of sides of a polygon, that is another centroid, and the correspondence of the number of turns of these centroids. The length of the centroid element is equal to the length of the side of the polygon. The number of elements of a centroid can be any integer, starting with one. Increasing the number of curvilinear centroid elements and the number of sides of a polygon increases the rotation uniformity of one centroid with respect to another.

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