1 Introduction

Sequences of the form $\left(\lfloor j\alpha \rfloor\right)_{j\in\mathbb{N}}$ for $\alpha > 1$, now known as Beatty sequences, were first studied in the context of the famous problem of covering the set of positive integers by disjoint sequences [1]. Further results in the direction of so-called disjoint covering systems are due to [2], [3], [4] and others. Other aspects of Beatty sequences were then studied, such as their generation using graphs [5], their relation to generating functions [6], [7], their substitution invariance [8], [9], etc. A good source of references on Beatty sequences and other related problems can be found in [10], [11].

$$\beta^i = G_i\beta - G_{i-1} \text{ and } \beta'^i = G_i\beta' - G_{i-1} \quad (5)$$

The proof we give here is based on the algebraic expression for one-dimensional cut-and-project sets [14]. Let V_1, V_2 be straight lines in $\mathbb{R}^2$ determined by vectors $(\beta, -1)$ and $(\beta', -1)$, respectively. The projection of the square lattice $\mathbb{Z}^2$ on the line V_1 along the direction of V_2 is given by

$$(a,b) = (a + b\beta')\vec{x}_1 + (a + b\beta)\vec{x}_2, \text{ for } (a,b) \in \mathbb{Z}^2,$$

where $\vec{x}_1 = \frac{1}{\beta - \beta'}(\beta, -1)$ and $\vec{x}_2 = \frac{1}{\beta' - \beta}(\beta', -1)$. For the description of the projection of $\mathbb{Z}^2$ on V_1 it suffices to consider the set

$$\mathbb{Z}[\beta'] := \{a + b\beta' | a, b \in \mathbb{Z}\}$$

The integral basis of this free abelian group is $(1, \beta')$, and thus every element x of $\mathbb{Z}[\beta']$ has a unique expression in this base. We will say that a is the rational part of $x = a + b\beta'$ and b is its irrational part. Since β' is a quadratic unit, $\mathbb{Z}[\beta']$ is a ring and, moreover, it satisfies

$$\beta'\mathbb{Z}[\beta'] = \mathbb{Z}[\beta'] \quad (6)$$

A cut-and-project set is the set of projections of points of $\mathbb{Z}^2$ to V_1, that are found in a strip of given bounded width, parallel to the straight line V_1. Formally, for a bounded interval Ω we define

$$\Sigma(\Omega) = \{a + b\beta' | a, b \in \mathbb{Z}, a + b\beta \in \Omega\}$$

Note that $a + b\beta'$ corresponds to the projection of the point (a, b) to the straight line V_1 along V_2, whereas $a + b\beta$ corresponds to the projection of the same lattice point to V_2 along V_1.

Among the simple properties of the cut-and-project sets that we use here are

$$\Sigma(\Omega - 1) = -1 + \Sigma(\Omega), \quad \beta'\Sigma(\Omega) = \Sigma(\beta\Omega),$$

where the latter is a consequence of (6). If the interval Ω is of unit length, one can derive directly from the definition a simpler expression for $\Sigma(\Omega)$. In particular, we have

$$\Sigma[0,1) = \{a + b\beta' | a + b\beta \in [0,1)\} = \{b\beta' - \lfloor b\beta \rfloor | b \in \mathbb{Z}\}, \quad (7)$$

where we use that the condition $0 \le a + b\beta < 1$ is satisfied if and only if $a = \lceil -b\beta \rceil = -\lfloor b\beta \rfloor$.

Let us mention that the above properties of one-dimensional cut-and-project sets, and many others, are explained in the review article [14].

3 Self-matching property of the graph $\lfloor j\beta \rfloor$ against j

An important role in the study of the self-matching properties of the graph $\lfloor j\beta \rfloor$ against j is played by the generalized Fibonacci sequence $(G_i)_{i\in\mathbb{N}}$, defined by (2) and (4), respectively. It turns out that shifting the argument j of the function $\lfloor j\beta \rfloor$ by the integer G_i results in shifting the value by G_{i-1}, with the exception of isolated mismatches with low frequency. The first proposition is an easy consequence of the expressions of β' as an element of the ring $\mathbb{Z}[\beta]$ in the integral basis $1, \beta$, given by (3) and (5).

Theorem 1

Let $\beta \in (0,1)$ satisfy $\beta^2 + m\beta = 1$ and let $(G_i)_{i=0}^{\infty}$ be defined by (2). Let $i \in \mathbb{N}$. Then for $j \in \mathbb{Z}$ we have

$$\lfloor \beta(j + G_i) \rfloor = \lfloor j\beta \rfloor + G_{i-1} + \varepsilon_i(j)$$

where $\varepsilon_i(j) \in \{0, (-1)^{i+1}\}$. The frequency of integers j for which the value $\varepsilon_i(j)$ is non-zero is equal to

$$\rho_i := \lim_{n\to\infty} \frac{\#\{j \in \mathbb{Z} | -n \le j \le n, \varepsilon_i(j) \ne 0\}}{2n+1} = \beta^i.$$

Proof. The first statement is trivial. For, we have

$$\begin{aligned} \varepsilon_i(j) &= \lfloor \beta(j + G_i) \rfloor - \lfloor j\beta \rfloor - G_{i-1} = \lfloor j\beta - \lfloor j\beta \rfloor + \beta G_i - G_{i-1} \rfloor \\ &= \lfloor j\beta - \lfloor j\beta \rfloor + (-1)^{i+1}\beta^i \rfloor \in \{0, (-1)^{i+1}\}. \end{aligned} \quad (8)$$

The frequency ρ_i is easily determined in the proof of Theorem 1. ❑

In the following theorem we determine the integers j for which $\varepsilon_i(j)$ is non-zero. From this, we easily derive the frequency of such mismatches.

Theorem 2

With the notation of Theorem 1, we have

$$\varepsilon_i(j) = \begin{cases} 0 & \text{if } j \notin U_i, \\ (-1)^{i+1} & \text{otherwise,} \end{cases}$$

where

$$U_i = \{kG_{i+1} + \lfloor k\beta \rfloor G_i | k \in \mathbb{Z}, k \ne 0\} \cup \left\{\frac{(-1)^{i-1}}{2} G_i\right\}.$$

Before starting the proof, let us mention that for i even, the set U_i can be written simply as

$$U_i = \{kG_{i+1} + \lfloor k\beta \rfloor G_i | k \in \mathbb{Z}\}.$$

For i odd, the element corresponding to $k = 0$ is equal to $-G_i$ instead of 0. The distinction according to the parity of i is necessary here, since unlike the paper [12], we determine the values of $\varepsilon_i(j)$ for $j \in \mathbb{Z}$, not only for.

Proof. It is convenient to distinguish two cases according to the parity of i.

- First let i be even. It is obvious from (8), that $\varepsilon_i(j) \in \{0, -1\}$ and

  $$\varepsilon_i(j) = -1 \quad \text{if and only if} \quad j\beta - \lfloor j\beta \rfloor \in [0, \beta^i). \quad (9)$$

  Let us denote by M the set of all such j,

  $$\begin{aligned} M &= \{j \in \mathbb{Z} | j\beta - \lfloor j\beta \rfloor \in [0, \beta^i)\} \\ &= \{j \in \mathbb{Z} | k + j\beta \in [0, \beta^i), \text{ for some } k \in \mathbb{Z}\} \end{aligned}$$

Therefore M is formed by the irrational parts of the elements of the set

$$\left\{k + j\beta' \middle| k + j\beta \in [0, \beta^i)\right\} = \Sigma[0, \beta^i) = \beta'^i \Sigma[0,1)$$
$$= (-\beta' G_i + G_{i-1})\left\{k\beta' - \lfloor k\beta \rfloor \middle| k \in \mathbb{Z}\right\},$$

where the last equality follows from (3) and (7). Separating the irrational part we obtain

$$M = \left\{k G_i\, m + k G_{i-1} + \lfloor k\beta \rfloor G_i \middle| k \in \mathbb{Z}\right\}$$
$$= \left\{G_i \lfloor k\beta \rfloor + k G_{i+1} \middle| k \in \mathbb{Z}\right\} = U_i,$$

where we have used the equations $\beta'^2 + m\beta' = 1$ and $mG_i + G_{i-1} = G_{i+1}$.

- Now let i be odd. Then from (8), $\varepsilon_i(j) \in \{0, -1\}$ and

$$\varepsilon_i(j) = 1 \quad \text{if and only if} \quad j\beta - \lfloor j\beta \rfloor \in [1 - \beta^i, 1). \tag{10}$$

Let us denote by M the set of all such j,

$$M = \left\{j \in \mathbb{Z} \middle| j\beta - \lfloor j\beta \rfloor - 1 \in [-\beta^i, 0)\right\}$$
$$= \left\{j \in \mathbb{Z} \middle| k + j\beta \in [-\beta^i, 0), \text{ for some } k \in \mathbb{Z}\right\}.$$

Therefore M is formed by the irrational parts of elements of the set

$$\left\{k + j\beta' \middle| k + j\beta \in [-\beta^i, 0)\right\} = \Sigma[-\beta^i, 0) = \beta'^i \Sigma[-1, 0)$$
$$= \beta'^i (1 - \Sigma[0,1)) = (\beta' G_i - G_{i-1})\left\{k\beta' - \lfloor k\beta \rfloor - 1 \middle| k \in \mathbb{Z}\right\}.$$

Separating the irrational part we obtain

$$M = \left\{-k G_i\, m - k G_{i-1} - \lfloor k\beta \rfloor G_i - G_i \middle| k \in \mathbb{Z}\right\}$$
$$= \left\{-k G_{i+1} - G_i(\lfloor k\beta \rfloor + 1) \middle| k \in \mathbb{Z}\right\}$$
$$= \left\{k G_{i+1} + G_i(\lceil k\beta \rceil - 1) \middle| k \in \mathbb{Z}\right\} = U_i,$$

where we have used the equation

$$\beta'^2 + m\beta' = 1,\ mG_i + G_{i-1} = G_{i+1} \text{ and } -\lfloor k\beta \rfloor = \lceil k\beta \rceil.$$

Let us recall that the Weyl theorem [15] states that numbers of the form $j\alpha - \lfloor j\alpha \rfloor$, $j \in \mathbb{Z}$, are uniformly distributed in (0, 1) for every irrational α. Therefore the frequency of those $j \in \mathbb{Z}$ that satisfy $j\alpha - \lfloor j\alpha \rfloor \in I \subset (0,1)$ is equal to the length of the interval I. Therefore one can derive from (9) and (10) that the frequency of mismatches (non-zero values $\varepsilon_i(j)$) is equal to β^i, as stated by Theorem 1. ❑

If $\beta \in (0,1)$ is the quadratic unit satisfying $\beta^2 - m\beta = -1$, then the considerations are even simpler, because expression (5) does not depend on the parity of i. We state the result as the following theorem.

Theorem 3
Let $\beta \in (0,1)$ satisfy $\beta^2 - m\beta = -1$ and let $(G_i)_{i=0}^{\infty}$ be defined by (4). For $i \in \mathbb{N}$, put

$$V_i = \left\{k G_{i+1} - (\lfloor k\beta \rfloor + 1) G_i \middle| k \in \mathbb{Z}\right\}.$$

Then for $j \in \mathbb{Z}$ we have

$$\lfloor \beta(j + G_i) \rfloor = \lfloor j\beta \rfloor + G_{i-1} + \varepsilon_i(j),$$

where

$$\varepsilon_i(j) = \begin{cases} 0 & \text{if } j \notin V_i, \\ 1 & \text{otherwise.} \end{cases}$$

The density of the set U_i of mismatches is equal to β^i.

Proof. The proof follows the same lines as proofs of Theorems 1 and 2. ❑

4 Conclusions

One-dimensional cut-and-project sets can be constructed from $\mathbb{Z}^2$ for every choice of straight lines V_1, V_2, if they have irrational slopes. However, in our proof of the self-matching properties of the Beatty sequences we strongly use the algebraic ring structure of the set $\mathbb{Z}[\beta']$ and its scaling invariance with the factor β', namely $\beta'\mathbb{Z}[\beta'] = \mathbb{Z}[\beta']$. For this, β' must necessarily be a quadratic unit.

However, it is plausible that, even for other irrationals α, some self-matching property is displayed by the graph $\lfloor j\alpha \rfloor$ against j. To show that, other methods would be necessary.

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References

[1] Beatty, S.: *Amer. Math. Monthly*, Vol. **33** (1926), No. 2, p. 103–105.

[2] Fraenkel, A. S.: The Bracket Function and Complementary Sets of Integers. Canad. J. Math., 21, 1969, 6–27

[3] Graham, R. L.: Covering the Positive Integers by Disjoint Sets of the Form $\{\lfloor n\alpha + \beta \rfloor : n = 1, 2, \ldots\}$. *J. Combinatorial Theory Ser. A*, Vol. **15** (1973), p. 354–358.

[4] Tijdeman, R.: Exact Covers of Balanced Sequences and Fraenkel's Conjecture. In *Algebraic Number Theory and Diophantine Analysis* (Graz, 1998), Berlin: de Gruyter 2000, p. 467–483.

[5] de Bruijn, N. G.: Updown Generation of Beatty Sequences. *Nederl. Akad. Wetensch. Indag. Math.*, Vol. **51** (1989), p. 385–407.

[6] Komatsu, T.: A Certain Power Series Associated with a Beatty Sequence. *Acta Arith.*, Vol. **76** (1996), p. 109–129.

[7] O'Bryant, K.: A Generating Function Technique for Beatty Sequences and Other Step Sequences. *J. Number Theory*, Vol. **94** (2002), p. 299–319.

[8] Komatsu, T.: Substitution Invariant Inhomogeneous Beatty Sequences. *Tokyo Journal Math.*, Vol. **22** (1999), p. 235–243.

[9] Parvaix, B.: Substitution Invariant Sturmian Bisequences. *Thor. Nombres Bordeaux*, Vol. **11** (1999), p. 201–210.

[10] Brown, T.: Descriptions of the Characteristic Sequence of an Irrational. *Canad. Math. Bull.*, Vol. **36** (1993), p. 15–21.

[11] Stolarsky, K.: Beatty Sequences, Continued Fractions, and Certain Shift Operators. *Canad. Math. Bull.*, Vol. **19** (1976), p. 473–482.

[12] Bunder, M., Tognetti, K.: On the Self Matching Properties of $[j\tau]$. *Discr. Math.*, Vol. **241** (2001), p. 139–151.

[13] Shechtman, D., Blech, I., Gratias, D., Cahn, J. W.: Metallic Phase with Long-Range Orientational Order and no Translation Symmetry. *Phys. Rev. Lett.*, Vol. **53** (1984), p. 1951–1953.

[14] Gazeau, J. P., Masáková, Z., Pelantová, E.: Nested Quasicrystalline Discretization of the Line. In: *Physics and Number Theory* (Editor: L. Nyssen), Vol. 10 of *IRMA Lectures in Mathematics and Theoretical Physics*, Zürich, EMS 2006, p. 79–132.

[15] Weyl, H.: Über die Gleichverteilung von Zahlen mod. *Eins. Math. Ann.*, Vol. **77** (1916), p. 313–352.

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